Circles and Triangle:

Q) Diagram (i) shows the incircle of ΔABC with radius 'r' and three other circles inside the triangle, each touching the incircle and two sides of triangle. The radii of the circles nearest to vertices A, B and C are r_A, r_b *and* r_c respectively. Show that $r_A + r_b + r_c \geq r$ with equality occurring when ΔABC is equilateral.

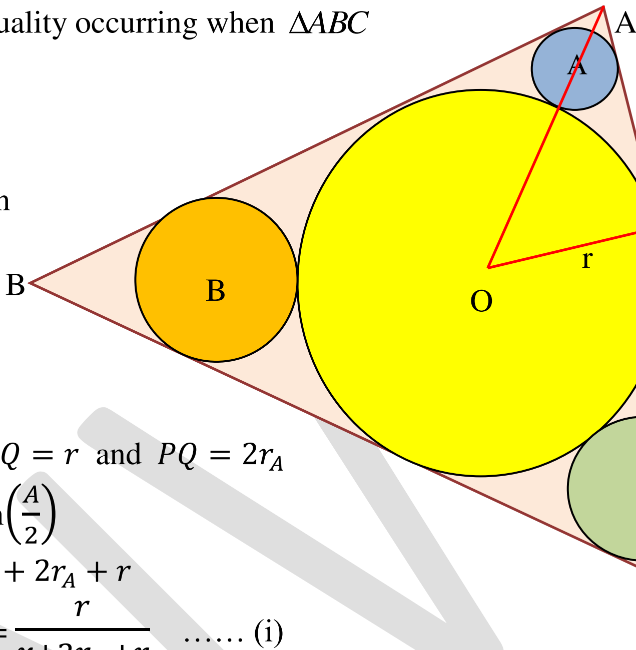
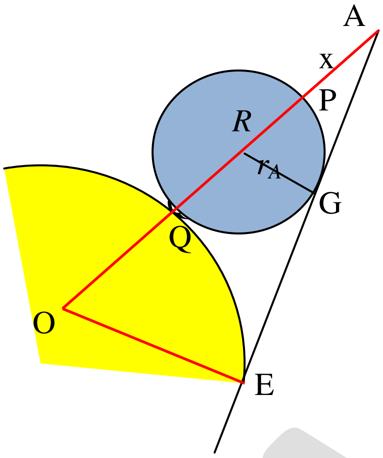


Diagram (i)

Solution: We consider the Diagram (ii) as shown



$OE = OQ = r$ and $PQ = 2r_A$

$\frac{OE}{OA} = \sin\left(\frac{A}{2}\right)$

$OA = x + 2r_A + r$

$\sin\left(\frac{A}{2}\right) = \frac{r}{x+2r_A+r}$ …… (i)

ΔARG and ΔAOE are similar

$\frac{r_A}{r} = \frac{AR}{AO} = \frac{x+r_A}{x+2r_A+r}$ …… (ii)

Eliminating 'x' in (i) and (ii), it can be shown that $\sin\left(\frac{A}{2}\right) = \frac{r-r_A}{r+r_A}$

⇨ $r_A = r\left(\frac{1-sin\left(\frac{A}{2}\right)}{1+sin\left(\frac{A}{2}\right)}\right)$ | Similarly $r_B = r\left(\frac{1-sin\left(\frac{B}{2}\right)}{1+sin\left(\frac{B}{2}\right)}\right)$ and $r_C = r\left(\frac{1-sin\left(\frac{C}{2}\right)}{1+sin\left(\frac{C}{2}\right)}\right)$

⇨ $r_A + r_B + r_C = r\left[\frac{1-sin\left(\frac{A}{2}\right)}{1+sin\left(\frac{A}{2}\right)} + \frac{1-sin\left(\frac{B}{2}\right)}{1+sin\left(\frac{B}{2}\right)} + \frac{1-sin\left(\frac{C}{2}\right)}{1+sin\left(\frac{C}{2}\right)}\right]$ …… (iii)

⇨ $r_A + r_B + r_C = r\left[\frac{\left(1-sin\left(\frac{A}{2}\right)\right)^2}{cos^2\left(\frac{A}{2}\right)} + \frac{\left(1-sin\left(\frac{B}{2}\right)\right)^2}{cos^2\left(\frac{B}{2}\right)} + \frac{\left(1-sin\left(\frac{C}{2}\right)\right)^2}{cos^2\left(\frac{C}{2}\right)}\right]$

The denominator of simplified expression is $cos^2\left(\frac{A}{2}\right)cos^2\left(\frac{B}{2}\right)cos^2\left(\frac{C}{2}\right)$

The expression has a minimum value when $cos\left(\frac{A}{2}\right)cos\left(\frac{B}{2}\right)cos\left(\frac{C}{2}\right)$ is maximum.

$cos\left(\frac{A}{2}\right)cos\left(\frac{B}{2}\right)cos\left(\frac{C}{2}\right)$ is maximum, when ΔABC is equilateral.

⇨ $A = B = C = 60^0$ and $sin\left(\frac{A}{2}\right) = sin\left(\frac{B}{2}\right) = sin\left(\frac{C}{2}\right) = \frac{1}{2}$

From (iii) $r_A + r_b + r_c \geq r$ with equality occurring when ΔABC is equilateral.