Multivariable Calculus by Edward and Penney

Q 15.5.28) Find the Centroid of a part of spherical surface with equation $\rho = a$ that lies within the cone $r = z$

Solution:

Concept used

The coordinates of the Centroid G$(\bar{x}, \bar{y}, \bar{z})$are computed as

$$\bar{x} = \frac{1}{m}\iint_S x\,\delta(x,y,z)\,dS, \quad \bar{y} = \frac{1}{m}\iint_S y\,\delta(x,y,z)\,dS, \quad \bar{z} = \frac{1}{m}\iint_S z\,\delta(x,y,z)\,dS$$

Where *m* is the mass of surface, given by

$$m = \iint_S \delta(x,y,z)\,dS$$

And

$\delta(x,y,z)$ is the mass density of the surface.

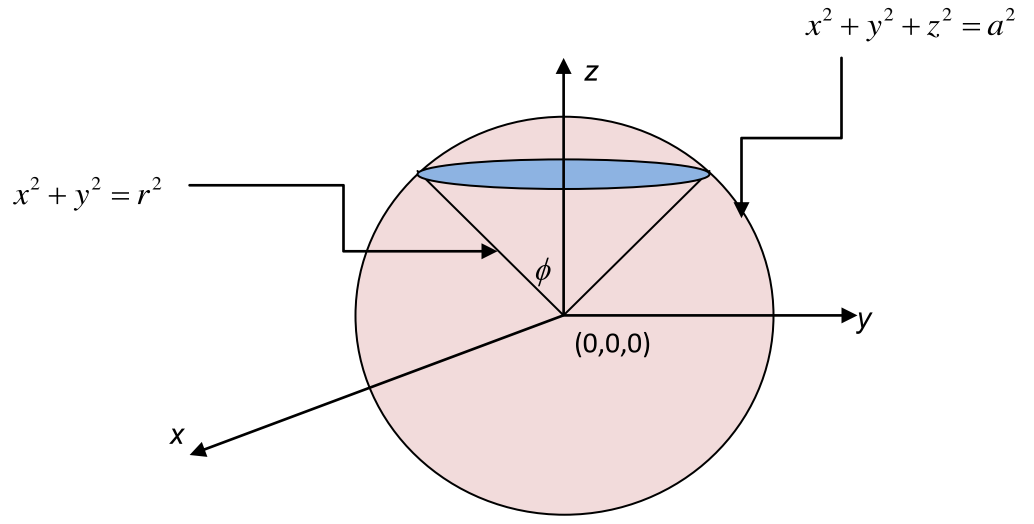
Equation of the sphere is

$\rho = a$

$\Rightarrow x^2 + y^2 + z^2 = a^2$

Equation of the cone is

$z = r \quad and \quad x^2 + y^2 = r^2$



Spherical Coordinates:

In a fixed frame of reference the Cartesian coordinates (x, y, z) are related to the spherical coordinates (ρ, θ, ϕ)through the relations

$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

$and\ z = \rho\cos\phi$

$dS = \rho^2 \sin\phi\, d\phi\, d\theta$

where $\rho > 0$, *is the distance of point from origin.*

θ *is longitude and* $0 \le \theta < 2\pi$

ϕ *is the latitude or polar angle and* $0 \le \phi < \pi$

For a sphere, $\rho = a$

$x^2 + y^2 + z^2 = a^2$

From the symmetry of the figure,

$\bar{x} = 0$

$\bar{y} = 0$

Also, equation of the cone

$z = r$

$x^2 + y^2 = r^2$

The semi – vertical angle of the cone, is

$$\tan\phi = \frac{r}{z}$$
$$= 1$$
$$\phi = \tan^{-1}(1)$$
$$== \frac{\pi}{4}$$

Therefore, mass is given by

$$m = \iint_S \delta(x,y,z)\,dS$$
$$= \int_0^{2\pi}\int_0^{\frac{\pi}{4}} \delta . a^2 \sin\phi\, d\phi\, d\theta$$
$$= (2\pi\delta\, a^2)\int_0^{\frac{\pi}{4}} \sin\phi\, d\phi$$
$$= (2\pi\delta\, a^2)[-\cos\phi]_0^{\frac{\pi}{4}}$$
$$= (2\pi\delta\, a^2)\left[\cos 0 - \cos\left(\frac{\pi}{4}\right)\right]$$
$$= (2\pi\delta\, a^2)\left[1 - \left(\frac{1}{\sqrt{2}}\right)\right]$$
$$= (2 - \sqrt{2})\pi\delta\, a^2$$

$$\bar{z} = \frac{1}{m}\iint_S z\,\delta(x,y,z)\,dS$$

$$= \frac{1}{(2-\sqrt{2})\pi\delta a^2}\int_0^{2\pi}\int_0^{\frac{\pi}{4}} z.\delta.a^2 \sin\phi\, d\phi\, d\theta$$

$$= \frac{1}{(2-\sqrt{2})\pi\delta a^2}\int_0^{2\pi}\int_0^{\frac{\pi}{4}} a\cos\phi.\delta.a^2 \sin\phi\, d\phi\, d\theta$$

$$= \frac{2\pi\delta a^3}{(2-\sqrt{2})\pi\delta a^2}\int_0^{\frac{\pi}{4}} \cos\phi \sin\phi\, d\phi$$

$$= \frac{2\pi\delta a^3}{(2-\sqrt{2})\pi\delta a^2}\left[\frac{\sin^2\phi}{2}\right]_0^{\frac{\pi}{4}}$$

$$= \frac{a}{(2-\sqrt{2})}\left[\left(\frac{1}{\sqrt{2}}\right)^2 - 0\right]$$

$$= \frac{a}{2(2-\sqrt{2})}$$

The coordinates of the Centroid are $G(\bar{x},\bar{y},\bar{z}) \equiv G\left(0,0,\frac{a}{2(2-\sqrt{2})}\right)$